Math 1320: Systems of Nonlinear Equations in Two Variables

What is a system of nonlinear equations? We have seen how to solve systems of linear equations in two and three variables with the substitution and addition methods. Another type of system we will solve are systems of nonlinear equations with two variables (x, y). A nonlinear equation is an equation that **cannot** be written in the form $Ax + By = C$. For example, one of the variables may have an exponent greater than one. In a system of nonlinear equations, we have two or more equations with at least one equation that is nonlinear. Consider the example below:

$$\begin{cases} x^2 = y - 1 \\ 4x - y = -1 \end{cases}$$

What is the solution of nonlinear systems? A solution to a nonlinear system in two variables is an ordered pair that makes all equations in the system true. Recall that systems of equations is an intersection problem, but with nonlinear equations, the graphs may be different. Because we are working with nonlinear equations, the graphs may be circles $(x^2 + y^2 = r^2)$, parabolas $(y = x^2)$, or anything other than two straight lines $(y = x)$. So it is possible that we may have no solutions, or one or more solutions to a nonlinear system.

How do we solve a system of nonlinear equations?

1. Substitution Method
   - The steps of the substitution method are the same as those used for solving linear systems in two variables.
   - This method is most useful if one of the given equations has an isolated variable, or if an equation has a variable that can be easily solved for.
2. Addition Method
   - The steps of the addition method are the same as those used for solving linear systems in two variables.
   - This method can be used for equations of the form $Ax^2 + By^2 = C$. If the equations are not given in this form, we may need to multiply either or both equations by appropriate numbers in order to eliminate the x or y terms with addition.

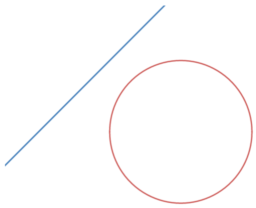
Algebra Warning: Recall some algebraic properties for solving equations:

1. When solving polynomial equations, we factor the polynomial and set each factor containing a variable equal to 0. Then solve for the variable.
2. When taking a square root to solve for a variable, it results in two solutions, the positive and negative value. [e.g. $x^2 = 4 \rightarrow x = \pm 2$]
3. Never divide by the variable to solve an equation. You will lose solutions.

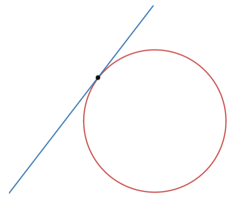
Example 1. Substitution Method

Solve by the substitution method: $\begin{cases} (x-2)^2 + (y+3)^2 = 4 & \text{(Eq. 1)} \\ -7 - y = -x & \text{(Eq. 2)} \end{cases}$

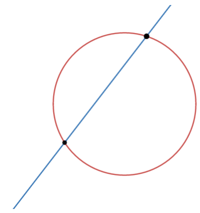
Before we apply the substitution method, let's think about the system graphically. Equation 1 is a circle and equation 2 is a line. The different possibilities of intersections of the two graphs and the corresponding number of solutions are below:



No solutions



1 Solution



2 Solutions

Step 1: Solve on of the equations for one variable in terms of the other.

Let's solve for x in equation 2:

$$\begin{aligned} -7 - y &= -x && \text{Eq. 2} \\ -(-7-y) &= x && \text{Multiply both sides by } -1 \\ 7 + y &= x && \text{Distribute} \end{aligned}$$

Step 2: Substitute the expression from step 1 into the other equation.

We need to substitute $7+y$ for x in equation 1:

$$\begin{aligned} (x-2)^2 + (y+3)^2 &= 4 && \text{Eq. 1} \\ (7+y-2)^2 + (y+3)^2 &= 4 && \text{Substitute } 7+y \text{ for } x \end{aligned}$$

Now we have an equation in terms of y only.

Step 3: Solve the resulting equation containing one variable.

$$\begin{aligned} (7+y-2)^2 + (y+3)^2 &= 4 && \rightarrow \text{Equation from step 2} \\ (5+y)^2 + (y+3)^2 &= 4 && \rightarrow \text{Combine like terms in parentheses} \\ 25 + 5y + 5y + y^2 + y^2 + 3y + 3y + 9 &= 4 && \rightarrow \text{Multiply: } (5+y)^2 = (5+y)(5+y) \\ & && \quad \text{and } (y+3)^2 = (y+3)(y+3) \\ 2y^2 + 16y + 34 &= 4 && \rightarrow \text{Combine like terms} \\ 2y^2 + 16y + 30 &= 0 && \rightarrow \text{Subtract 4 from both sides} \\ 2(y^2 + 8y + 15) &= 0 && \rightarrow \text{Factor out 2} \\ 2(y+3)(y+5) &= 0 && \rightarrow \text{Factor completely} \\ y+5 = 0 \qquad y+3 &= 0 && \rightarrow \text{Set each factor with } y \text{ equal to 0} \\ y = -5 \qquad y &= -3 && \rightarrow \text{Solve for } y \end{aligned}$$

Step 4: Back-substitute the obtained values into the equation from step 1.

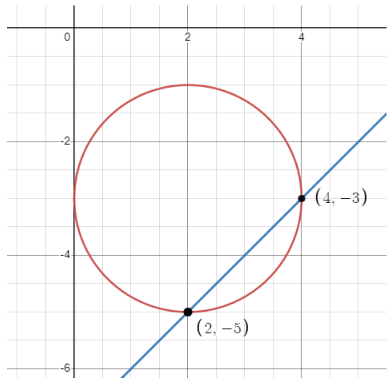
Now, we substitute -5 and -3 for y in the equation $x = 7 + y$ from step 1:

If $y = -5$: $x = 7 + (-5) = 2$, so $(2, -5)$ is a solution.

If $y = -3$: $x = 7 + (-3) = 4$, so $(4, -3)$ is a solution.

Step 5: Check the proposed solutions in both of the system's given equations.

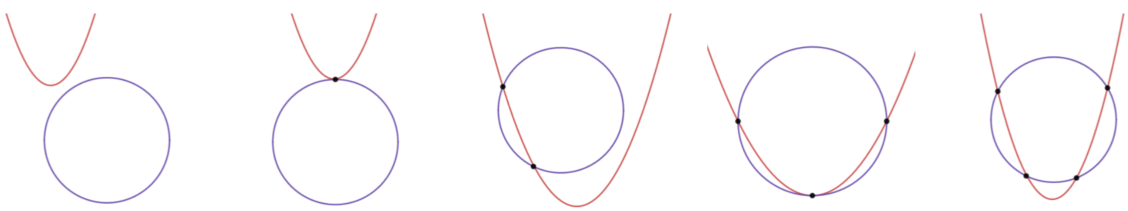
Does $(2, -5)$ and $(4, -3)$ make the equations in our original system true? Yes, by substituting the values for x and y into each of the two original equations, we get two true statements. We may also check these solutions graphically. From the graph below, we should have 2 solutions:



Example 2. Addition Method

Solve the system by the addition method: $\begin{cases} y = \frac{1}{2}x^2 - 2 & \text{(Eq.1)} \\ x^2 + y^2 = 4 & \text{(Eq.2)} \end{cases}$

Before we apply the addition method, let's think about the system graphically. Equation 1 is a parabola and equation 2 is a circle. The different possibilities of intersections of the two graphs and the corresponding number of solutions are below:



No Solution | 1 Solution | 2 Solutions | 3 Solutions | 4 Solutions

Although we could use the substitution method since equation 1 is already written in terms of x, substituting $\frac{1}{2}x^2 - 2$ for y in $x^2 + y^2 = 4$ would result in a fourth degree equation to solve. We don't know how to solve fourth degree polynomials, so let's apply the addition method:

Step 1: Write both equations in the form $Ax^2 + By^2 = C$.

$$-\frac{1}{2}x^2 + y = -2 \qquad \text{Subtract } -\frac{1}{2}x^2 \text{ from both sides}$$

$$x^2 + y^2 = 4 \qquad \text{No Change}$$

Step 2: If necessary, multiply either equation or both equations by appropriate numbers so that the sum of the x^2-coefficients or the sum of the y^2-coefficients is 0.

$$-\frac{1}{2}x^2 + y = -2 \qquad \text{No Change} \longrightarrow \qquad -\frac{1}{2}x^2 + \quad y = -2$$

$$x^2 + y^2 = 4 \qquad \text{Multiply by } \frac{1}{2} \longrightarrow \qquad \frac{1}{2}x^2 + \frac{1}{2}y^2 = 2$$

Step 3 and 4: Add equations and solve for the remaining variable.

$$
\begin{array}{rl}
-\frac{1}{2}x^2 + \quad y & = -2 \\
\frac{1}{2}x^2 + \frac{1}{2}y^2 & = 2 \\
\hline
\text{Add:} \quad \frac{1}{2}y^2 + y & = 0
\end{array}
$$

$$
\begin{array}{lll}
y(\frac{1}{2}y+1)=0 & & \rightarrow \text{ Factor out } y \text{ from both terms} \\
y=0 & \frac{1}{2}y+1=0 & \rightarrow \text{ Set both factors equal to } 0 \\
y=0 & \quad y=-2 & \rightarrow \text{ Solve for } y
\end{array}
$$

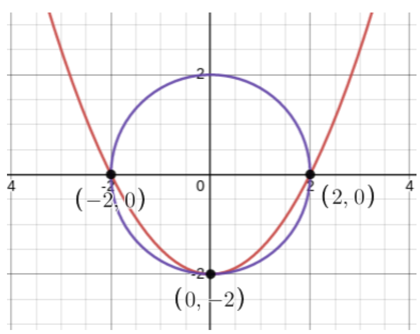
Step 5: Back-substitute and find the values for the other variable. Now, we can substitute 0 and -2 for y into either one of the original equations. Let's use $x^2 + y^2 = 4$, equation 2:

$x^2 + y^2 = 4$	Equation 2	$x^2 + y^2 = 4$	Equation 2
$x^2 + (0)^2 = 4$	Replace y with 0	$x^2 + (-2)^2 = 4$	Replace y with -2
$x^2 = 4$	Evaluate 0^2	$x^2 + 4 = 4$	Evaluate $(-2)^2$
$x = \pm 2$	Use square root property	$x^2 = 0$	Subtract 4 from both sides
		$x = 0$	Use square root property

$(0, 2)$, $(0, -2)$, $(-2, 0)$ are solutions.

Step 5: Check the proposed solutions in both of the system's given equations.

Does $(0, 2)$, $(0, -2)$, and $(-2, 0)$ make the equations in our original system true? Yes, by substituting the values for x and y into each of the two original equations, we get two true statements. We may also check these solutions graphically. From the graph below, we should have 3 solutions:



Practice Problems

Solve each system by the method of your choice.

1. $\begin{cases} -2x + y = -4 \\ -x^2 + 2y = -8 \end{cases}$ $\quad [(0, -4),\ (4, 4)]$

2. $\begin{cases} x^2 + y^2 = 25 \\ 2x + 4y = 10 \end{cases}$ $\quad [(-3, 4),\ (5, 0)]$